ON TWO CLASSES OF PLANE EXTREMAL MOTIONS OF A ROCKET IN VACUUM

(O DVUKH KLASSAKH PLOSKIKH ÈKSTREMAL'NYKH DVIZHENIIA RAKETY V PUSTOTE)

PMM Vol.24, No.2, 1960, pp. 303-308

Iu.A. GORELOV
(Moscow)

(Received 6 May 1959)

A study of the conditions is given, the fulfilment of which will secure a motion of rockets along curvilinear trajectories which is extremal in time and expenditure of mass.

The rocket is assumed to be an ideally controlled body, i.e. it may occupy instantaneously the necessary angular orientation in space, rotating about its longitudinal or transverse axes which pass through its center of gravity. This assumption permits a formulation of the problem, based only on the force equations of motion, assuming the moment equations always to be satisfied.

The motion is considered for an active segment of a trajectory, i.e. the mass of the rocket is assumed to vary in time.

Within the framework of the indicated hypotheses, the variational problem is formulated, regarding the determination of the characteristics of extreme manoeuvres [turns, rotations] in time of rockets by given angles, for given initial and final velocities of motion, initial and final weight of rockets. The solution of the problem is sought for plane motions in horizontal and vertical planes separately, under the assumption of absence of the influence of aerodynamic forces.

1. If the trajectory of the motion of a rocket, executing a controlled manoeuvre, is strictly in the horizontal plane, then, projecting the forces on the tangential and normal directions (Fig. 1), we have the two basic equations

$$m\frac{dV}{dt} = T\cos\alpha \qquad (1.1)$$

$$mV\frac{d\gamma}{dt} = \sqrt{T^2\sin^2\alpha - (mg)^2} \qquad (1.2)$$

Here I is the thrust of the rocket engine, a is the angle of attack (the angle between the longitudinal axis of the body and the tangent to the trajectory), γ is the angle of rotation of the trajectory, measured from the initial position. Assuming the mass m of the rocket to vary, we will write the equation of change of mass in the form

$$-\frac{dm}{dt} = Tq \tag{1.3}$$

where q is the expenditure of mass per sec per unit thrust.

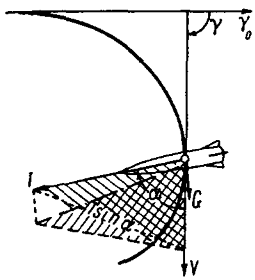

FIG. 1.

Equations (1.1)-(1.3) contain six variable quantities: m, T, t, a, γ, V.

We obtain from (1.1) and (1.3)

$$-m\frac{dV}{dm} = \frac{\cos\alpha}{q}$$

Introducing, as usual, the variable $\phi = q^{-1}\log m$, we obtain

$$\frac{dV}{d\varphi} + \cos\alpha = 0 \tag{1.4}$$

Next, we substitute in (1.2) for the variables T and a from (1.3) and (1.4):

$$(mV)^2\left(\frac{d\gamma}{dt}\right)^2 = \frac{1}{q^2}\left(\frac{dm}{dt}\right)^2\left[1-\left(\frac{dV}{d\varphi}\right)^2\right] - (mg)^2$$

or

$$\left(\frac{d\gamma}{dt}\right)^2\left[\left(\frac{d\varphi}{d\gamma}\right)^2 - V^2 - \left(\frac{dV}{d\gamma}\right)^2\right] = g^2 \tag{1.5}$$

Equation (1.5) allows us, by taking the quantity γ as basic independent variable, to obtain the expression for the time of the manoeuvre:

$$\tau = \pm\frac{1}{g}\int_{\gamma_0}^{\gamma_k} F(V, V', \varphi')\,d\gamma \qquad \left(F = \sqrt{\varphi'^2 - V^2 - V'^2}\right) \tag{1.6}$$

where primes denote total derivatives of the unknown functions with respect to γ. The plus sign corresponds to motion for which the angle of rotation increases monotonically with time, the minus sign to motion for which it decreases. Since the change of mass of the rocket in accordance with the material balance equation (1.3) is related simply to the thrust, and the velocity of motion along the trajectory is connected with the angle of attack (1.4), then simultaneous determination of the extremal dependence between the velocity and the mass of a rocket will lead to the law of change of the thrust and the angle of attack during the manoeuvre. Thus, the determination of the necessary conditions securing motion of rockets along curvilinear trajectories, extremal in time, is reduced to the construction of the Euler equation for the function $F(V, V', \phi')$, and the problem of the determination of the time of the manoeuvre is reduced to finding the extremum of the functional (1.6) for the usual boundary conditions for the functions V and ϕ, i.e.

$$V = V_0, \quad \varphi = \varphi_0, \quad \text{for } \gamma = \gamma_0, \qquad V = V_k, \quad \varphi = \varphi_k \quad \text{for } \gamma = \gamma_k$$

2. A necessary condition for the existence of an extremum of the functional (1.6) is the fulfilment of the Euler equation for the function F:

$$\frac{d}{d\gamma}\frac{\partial F}{\partial \varphi'} = 0, \qquad \frac{d}{d\gamma}\frac{\partial F}{\partial V'} - \frac{\partial F}{\partial V} = 0 \tag{2.1}$$

Substituting the expression for the derivatives of F into (2.1), we obtain two second-order equations which, after some manipulations, may be reduced to the form

$$\frac{\varphi''}{\varphi'} = \frac{VV' + V'V''}{V^2 + V'^2}, \qquad \frac{\varphi''}{\varphi'} = \frac{V'' - V}{V'} \tag{2.2}$$

The solutions of these equations will now be found.

Comparing the right-hand sides of Equations (2.2) we obtain the differential equation

$$V''V - 2V'^2 - V^2 = 0 \tag{2.3}$$

Lowering its order by a suitable substitution of variables and integrating the resulting linear equation, one obtains a first integral in the form

$$V' = \pm V\sqrt{C_1^2V^2 - 1} \tag{2.4}$$

and, consequently, also the final dependence of the velocity of flight on the angle of rotation of the trajectory

$$\Delta\gamma = \gamma_k - \gamma_0 = \pm\left[\sec^{-1}\left(\frac{V}{V_0}C_1\right) - \sec^{-1} C_1\right] = \pm\left[\sec^{-1}\left(\frac{V}{V_0}C_1\right) - \gamma^*\right] \tag{2.5}$$

Since in the horizontal plane all initial values γ_0 are equivalent, one may, including γ in γ^*, assume $\Delta\gamma = \gamma_k = \gamma$. The family of extremal curves $V/V_0 = f(\gamma)$ is obtained in the form

$$\frac{V}{V_0} = \frac{1}{C_1 \cos(\gamma - \gamma^*)} \tag{2.6}$$

It depends on the two parameters C and γ^*, the values of which are determined by the given boundary conditions V_0 and V_k. In the curves of Fig. 2 depicting the change of the relative velocity, one of the parameters (γ^*) has already been excluded by use of the condition $V/V_0 = 1$ for $\gamma = \gamma_0$. Thus, for a given value of the relative final velocity one has still to determine the second parameter C_1. The quantity C_1 is readily determined for given values of $\Delta\gamma$ and V_k/V_0, the relative value of the final velocity (Fig. 2).

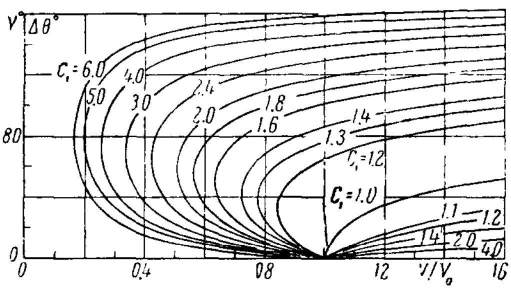

FIG. 2.

Using the above dependence $V/V_0 = f(\gamma, C_1)$, we will determine by integration of one of the equations (2.2) the change of mass or weight of the rocket during the manoeuvre:

$$\frac{G}{G_0} = \exp\left\{C_2 qg\left(\frac{V}{V_0}\,;\,C_1\right)\right\} \tag{2.7}$$

where the weight function

$$g = \frac{1}{C_1^2}\left(\sqrt{C_1^2\left(\frac{V}{V_0}\right)^2 - 1} - \sqrt{C_1^2 - 1}\right) \tag{2.8}$$

shown in Fig. 3, is determined by the condition $G/G_0 = 1$ for $\gamma = \gamma_0$. The second constant C_2 is obtained from the second boundary condition of the value of the final weight at the end of the manoeuvre.

The time of the completion of the extremal manoeuvre through a given angle is obtained by substitution of the relations (2.6) and (2.7) into the functional

$$\tau = \frac{V_0}{g}\sqrt{C_2^2 - C_1^2}\, g\left(\frac{V}{V_0}\,,\,C_1\right) \tag{2.9}$$

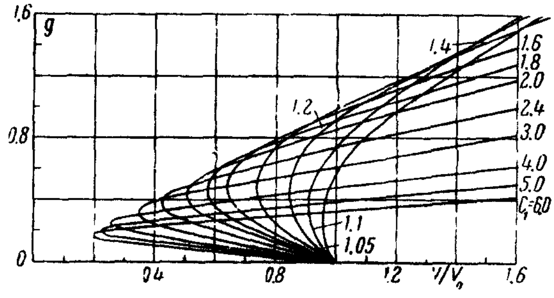

FIG. 3.

3. We will consider next the curvilinear motion of a rocket in a vertical plane. The basic equations in terms of the projections on the tangential and normal directions to the flight trajectory, together with the equation of mass balance, have the form

$$m\frac{dV}{dt} = T\cos\alpha - mg\sin\theta,\quad mV\frac{d\theta}{dt} = T\sin\alpha - mg\cos\theta,\quad -\frac{dm}{dt} = Tq \qquad (3.1)$$

We will transform as we did in Section 1. Squaring the equations (3.1), adding them and collecting terms containing the same powers of $d\theta/dt$, we obtain the quadratic of three terms

$$\left(\frac{d\theta}{dt}\right)^2\left\{\left(\frac{dV}{d\theta}\right)^2 - \left(\frac{d\varphi}{d\theta}\right)^2 + V^2\right\} + \left(\frac{d\theta}{dt}\right)2g\left\{\sin\theta\left(\frac{dV}{d\theta}\right) + \cos\theta V\right\} + g^2 = 0$$

the solution of which, for $d\theta/dt$, after removal of irrationality in the denominator, permits the writing-down of the functional for the time of manoeuvre in the following form:

$$\tau = \frac{1}{g}\int_{\theta_0}^{\theta_k} H(V, V', \varphi', \theta)\,d\theta \qquad (3.2)$$

where

$$H = (V\cos\theta + V'\sin\theta) \pm \sqrt{(V\cos\theta + V'\sin\theta)^2 + [\varphi'^2 - V^2 - V'^2]}$$

As has been shown in Section 1, the expression (3.2) is the principal formulation of the same variational problem for trajectories of curvilinear motion, only that now they are in the vertical plane. The structure of the function H, in comparison with F, is complicated by the presence of the trigonometric function of the independent variable. At the same time, the three terms $\phi'^2 - V^2 - V'^2$ of the function F are also present. This indicates beforehand that the final results will contain the solution which applied for the consideration of the horizontal motions. Thus, eliminating the variables V and ϕ from the Euler equation for the function H, we obtain an identity into which enters the left-hand side of (2.3), determining the extremal law of change of the velocity for an angle of rotation in the horizontal plane. Consequently, the solutions of this equation, satisfying the Euler equation, also in the given case single out the particular class of extremal relations $V = f(\theta)$ which, by analogy with (2.6), may be written in the form

$$\frac{V}{V_0} = \frac{1}{C_1 \cos(\theta - \theta^*)} \tag{3.3}$$

where the quantity θ^*, as before γ^*, contains the initial angle of rotation θ_0 which must be kept in mind when determining the parameter θ^* from the conditions $V/V_0 = 1$ for $\theta = \theta_0$. In addition, since the expression for the function H again does not explicitly contain the variable ϕ, the Euler equations will have the form (2.1). The first of these, after evaluation of the derivatives of H, gives the first integral

$$\varphi' = C_3 (V \sin\theta - V' \cos\theta) \tag{3.4}$$

Substituting (3.3) in (3.4) and integrating with respect to the angle of rotation, we find the change of mass or weight of the rocket:

$$\frac{G}{G_0} = \exp\{C_3 q V_0 g_{(\theta,\, \theta_0,\, \theta^*,\, C_1)}\} \tag{3.5}$$

where the weight function is

$$g_{(\theta,\, \theta_0,\, \theta^*,\, C_1)} = \frac{\sin\theta^*}{C_1}[\tan(\theta - \theta^*) - \tan(\theta_0 - \theta^*)] \tag{3.6}$$

The time of completion of the extremal manoeuvre by the angle $\Delta\theta = \theta_k - \theta_0$ may be determined from the formula

$$\tau = \frac{V_0}{g}(\cos\theta^* \pm \sin\theta^* \sqrt{C_3{}^2 - 1})\, g_{(\theta,\, \theta_0,\, \theta^*,\, C_1)} \tag{3.7}$$

Thus, by the subsequent determination of the constants C_1 and C_2 or C_1 and C_3 from the four boundary conditions V_0, V_k, and G_0, G_k one may produce, in correspondence with any arbitrary trajectory (lying in one of the planes under consideration), the extremal trajectory with such boundary conditions. Completing the computations, it is easily verified that the extremal dependences for the angles of attack and controlled thrust realize a maximum of the functionals (1.6) and (3.2), i.e. they single out the trajectories with maximum duration of flight.

4. We will now state a most general feature characterizing a given class of extremal motions.

The ratio of thrust to weight of rockets during the process of motion along extremal trajectories is constant.

This result is general for motions in horizontal and vertical planes. It may be deduced by substituting the extremal laws (2.7) and (2.9) or (3.3) and (3.4), taken in differential form, in the original equations of change of mass (1.3).

For motion in horizontal planes

$$T^{\circ} = \frac{T}{mg} = -\frac{1}{g}\frac{d\varphi}{d\gamma}\frac{d\gamma}{dt} = \frac{-C_2}{V_0\sqrt{C_2^2 - C_1^2}} = \text{const} \tag{4.1}$$

For motion in vertical planes

$$T^{\circ} = \frac{-1}{g}\frac{d\varphi}{d\theta}\frac{d\theta}{dt} = \frac{C_3}{\dfrac{(V\cos\theta + V'\sin\theta)}{(V\sin\theta - V'\cos\theta)} \pm \sqrt{C_3^2 - 1}} = \frac{C_3}{\cot\theta^* \pm \sqrt{C_3^2 - 1}} = \text{const} \tag{4.2}$$

The condition T_0 = const characterizes in the given problem a class of motions of bodies of variable mass, when the acting reactions change proportionally to the change of mass of the body. Thus, the programming of the thrust would also be connected with the change of the weight of the rocket. The trajectory of motion and all its characteristics for fulfilment of this condition remain the same as for the motion of a body of constant mass with constant reaction. The angular orientation of the rockets in space for motion along extremal trajectories does not change:

$$\frac{d\alpha}{d\gamma} \approx \frac{d\alpha}{d\theta_1} = -1 \tag{4.3}$$

For motion in vertical planes this feature exactly characterizes the law of change of the angle of attack; for motion in horizontal planes the sign of the approximate equality is explained by the disposition of the angles α and γ in different planes.

The invariability of the angular orientation of rockets during motion along extremal trajectories follows from the law of change of the angle of attack in terms of the angle of rotation θ or γ. We will reset the first equations (3.1) in the form

$$T^{\circ}\cos\alpha - \sin\theta = \frac{1}{g}V'\frac{d\theta}{dt}, \qquad T^{\circ}\sin\alpha - \cos\theta = \frac{1}{g}V\frac{d\theta}{dt} \tag{4.4}$$

Using the extremal laws

$$\frac{d\theta}{dt} = \frac{-gC_1\cos^2(\theta - \theta^*)}{\{\cos\theta^* \pm \sin\theta^*\sqrt{C_3^2 - 1}\}}, \qquad V = \frac{1}{C_1\cos(\theta - \theta^*)}$$

we obtain from (3.3)

$$(T^{\circ}\cos\alpha - \sin\theta)^2 + (T^{\circ}\sin\alpha - \cos\theta)^2 = \{\cos\theta^* \pm \sin\theta^*\sqrt{C_3^2 - 1}\}^{-2}$$

or

$$1 + T^{\circ 2} - \{\cos\theta^* \pm \sin\theta^* \sqrt{C_3^2 - 1}\}^{-2} = 2T^\circ \sin(\alpha + \theta) \qquad (4.5)$$

Since for motion along the extremal $T^0 =$ const, it follows from (4.5) that $(\alpha + \theta) =$ const or $d\alpha/d\theta = -1$.

For study of motions, extremal in time in horizontal planes we obtain by analogous transformations the following link between the angle of attack and the angle of rotation:

$$\cos\alpha = \frac{\sqrt{T^{\circ 2} - 1}}{T^\circ} \sin(\gamma - \gamma^*) \quad \text{or} \quad \frac{d\alpha}{d\gamma} = -\frac{\sqrt{T^{\circ 2}\sin^2\alpha - 1}}{T^\circ \sin\alpha} \qquad (4.6)$$

It is seen from (4.6) that for large values of T^0, practically for $T^0 > 4$, the quantity $d\alpha/d\gamma \approx -1.0$. Since the angle of attack is between the directions of the longitudinal axis of the rocket and of the tangent to the trajectory, then for a change of the angle of rotation there follows a simultaneous change of the angle of attack, so that an increase of the angle of rotation leads to a decrease of the angle of attack by the same amount. Consequently, fulfilment of the condition $d\alpha/d\theta = -1$ corresponds to preservation of the angular orientation of the rocket in space, constant with respect to a fixed observer. The extremals found for the time of motion secure simultaneously minimal expenditure of fuel.

We will solve the original differential equation (1.5) for the derivative $d\phi/d\gamma$ and form a new functional, expressing the change of mass or weight of the rocket for rotation by a given angle $\Delta\gamma = \gamma_k - \gamma_0$. It follows from (1.5) that

$$\varphi'^2 = \tau'^2 + V^2 + V'^2 \qquad \left(\tau' = \frac{1}{g}\frac{dt}{d\gamma}\right)$$

since $\phi = 1/q \log m$, $\phi_k - \phi_0 = 1/q \log G_k/G_0$, one has

$$\frac{G_k}{G_0} = \exp\left\{-q \int_{\gamma_0}^{\gamma_k} \sqrt{\tau'^2 + V^2 + V'^2}\, d\gamma\right\} \qquad (4.7)$$

Thus, the problem of finding the conditions guaranteeing the least consumption of fuel for a curvilinear manoeuvre of a rocket by a given angle leads to the determination of the extremes of the functional (4.7). It is readily verified that the extremal rules for the change of velocity and mass with the angle of rotation and consequently with angles of attack and thrust are found to be the same as in the case of the problem of extremum time of motion.

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Translated by J.R.M.R.